离散数学（011122）

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CAMEA 中国高质量MBA教育认证　AACSB ACCREDITED　EQUIS ACCREDITED

- **2.2.1 Equivalence Expressions and Equivalence Calculus**
  - Equivalence Expressions and Basic Equivalence Expressions
  - Truth Table Method and Equivalence Calculus Method
- **2.2.2 Connective Complete Set**
  - Truth Functions
  - Connective Complete Set
  - NAND Connective, NOR Connective

Definition 2.11:

Let *A* and *B* be two propositional formulas. If the equivalence expression $A \leftrightarrow B$ is a tautology (Universally Valid Formula), then *A* and *B* are said to be equivalent, denoted as $A \Leftrightarrow B$, and called an ***equivalence expression***.

Explanation:

(1) $\Leftrightarrow$ is the notation for equivalence, which is different from the equivalence connective $\leftrightarrow$.

(2) $A \Leftrightarrow B$ means that propositions *A* and *B* are either both "true" or both "false" under all possible assignments (i.e., they have the same truth table).

(3) Every propositional formula has infinitely many equivalent propositional formulas(e.g.,: $\neg\neg P$ is equivalent to *P*).

↳ Equivalence Expressions (cont.)

■ **Explanation:**

(3) Every propositional formula has infinitely many equivalent propositional formulas(e.g.,: $\neg\neg P$ is equivalent to P).

(4) In propositional logic, "Equivalence" is a more commonly used and clearer term used to indicate that two propositions have the same truth value under all possible conditions. This is similar to the concept of "Equality".

(5) There may be dummy variables in A or B.

For example, in $(p\rightarrow q) \leftrightarrow ((\neg p\vee q)\vee(\neg r\wedge r))$, r is a dummy variable in the left-hand formula.

The value of a dummy variable does not affect the truth value of the propositional formula.

e.g. Example： Judge $\neg(p\vee q)$ and $\neg p\wedge\neg q$ is equal or not.

Solve：

p	q	$\neg p$	$\neg q$	$p\vee q$	$\neg(p\vee q)$	$\neg p\wedge\neg q$	$\neg(p\vee q)\leftrightarrow(\neg p\wedge\neg q)$
0	0	1	1	0	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	1	0	0	1
1	1	0	0	1	0	0	1

Conclusion: $\neg(p\vee q) \Leftrightarrow (\neg p\wedge\neg q)$

↳ Determine Equivalence Expression Using a Truth Table (cont.)

e.g. **Example:** Determine the equivalence relationship among the following three formulas:

$$p\to(q\to r),\ (p\to q)\to r,\ (p\wedge q)\to r$$

Solve:

p	q	r	$p\to(q\to r)$	$(p\to q)\to r$	$(p\wedge q)\to r$
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Conclusion:

$$p\to(q\to r) \Leftrightarrow (p\wedge q)\to r$$

↳ Basic Equivalence Expressions

- **Double Negation Law:** $\neg\neg A \Leftrightarrow A$
- **Idempotent Law:** $A \vee A \Leftrightarrow A$, $A \wedge A \Leftrightarrow A$
- **Commutative Law:** $A \vee B \Leftrightarrow B \vee A$, $A \wedge B \Leftrightarrow B \wedge A$
- **Associative Law:** $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$
  $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
- **Distributive Law:** $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
  $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
- **De Morgan's Laws:** $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$
  $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$
- **Absorption Law:** $A \vee (A \wedge B) \Leftrightarrow A$
  $A \wedge (A \vee B) \Leftrightarrow A$

↳ Basic Equivalence Expressions (cont.)

- Zero Law: $A \vee 1 \Leftrightarrow 1, \quad A \wedge 0 \Leftrightarrow 0$
- Identity Law: $A \vee 0 \Leftrightarrow A, \quad A \wedge 1 \Leftrightarrow A$
- Law of the Excluded Middle: $A \vee \neg A \Leftrightarrow 1$
- Law of Contradiction: $A \wedge \neg A \Leftrightarrow 0$
- Implication Equivalence: $A \rightarrow B \Leftrightarrow \neg A \vee B$
- Biconditional Equivalence: $A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$
- Contraposition: $A \rightarrow B \Leftrightarrow \neg B \rightarrow \neg A$
- Negation of Equivalence: $A \leftrightarrow B \Leftrightarrow \neg A \leftrightarrow \neg B$
- Reductio ad Absurdum (Proof by Contradiction):

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \Leftrightarrow \neg A$$

- **Equivalence Calculus**: The process of deriving new equivalences from known equivalences.
- **Substitution Rule** **If** $A \Leftrightarrow B$, **then** $\Phi(B) \Leftrightarrow \Phi(A)$

e.g. **Example**: To prove $p \to (q \to r) \Leftrightarrow (p \land q) \to r$

Proof: $p \to (q \to r)$

$\Leftrightarrow \neg p \lor (\neg q \lor r)$ (Implication Equivalence)

$\Leftrightarrow (\neg p \lor \neg q) \lor r$ (Associative Law)

$\Leftrightarrow \neg (p \land q) \lor r$ (De Morgan's Law)

$\Leftrightarrow (p \land q) \to r$ (Implication Equivalence)

↳ Methods to Prove the Non-Equivalence of Two Formulas

e.g. **Example: Proof:** $p\to(q\to r) \not\Leftrightarrow (p\to q)\to r$

- Equivalence calculus **cannot directly prove** that two formulas are not equivalent.
- The fundamental idea to prove ***non-equivalence*** is to find an assignment that makes one formula **true** while making the other **false**.
- **Key Approaches:**
  - **Truth Table Method**
  - **Observation Method:** It is easy to see that the assignment (p, q, r) = (0,0,0) makes the left formula **true** and the right formula **false**.
  - **Simplification and Observation:**

e.g. **Example:** Use **equivalence calculus** to determine the **type** of the following formula:

(1) $q \wedge \neg(p \rightarrow q)$

Solution: $q \wedge \neg(p \rightarrow q)$

$\Leftrightarrow q \wedge \neg(\neg p \vee q)$ (Implication Equivalence)

$\Leftrightarrow q \wedge (p \wedge \neg q)$ (De Morgan's Law)

$\Leftrightarrow p \wedge (q \wedge \neg q)$ (Commutative and Associative Laws)

$\Leftrightarrow p \wedge 0$ (Law of Contradiction)

$\Leftrightarrow 0$ (Zero Law)

Thus, the formula is a ***contradiction***.

e.g. **Example：** Use **equivalence calculus** to determine the **type** of the following formula:

(2) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Solution: $(p \to q) \leftrightarrow (\neg q \to \neg p)$

$\Leftrightarrow (\neg p \vee q) \leftrightarrow (q \vee \neg p)$ （Implication Equivalence）

$\Leftrightarrow (\neg p \vee q) \leftrightarrow (\neg p \vee q)$ （Commutative Law）

$\Leftrightarrow 1$

this formula is a **tautology** (always true).

(3) $((p\wedge q)\vee(p\wedge\neg q))\wedge r$

Solution: $((p\wedge q)\vee(p\wedge\neg q))\wedge r$

$\Leftrightarrow (p\wedge(q\vee\neg q))\wedge r$ （Distributive Law)）

$\Leftrightarrow p\wedge 1\wedge r$ （Law of Excluded Middle）

$\Leftrightarrow p\wedge r$ （Identity Law）

- This is a **satisfiable formula**, but not a **tautology**. For example:
- **101** is a **truth assignment** that makes it true.
- **000** is a **truth assignment** that makes it false.

■ **Summary:**

- A formula *A* is a ***contradiction*** if and only if A≡0.
- A formula *A* is a ***tautology*** if and only if A≡1.

- **Definition 2.12: $F:\{0,1\}^n \to \{0,1\}$ n-ary truth-value function.**
  - The **n** propositional variables can form 2^{2^n} truth-value functions (**n≥1**).
  - Each **propositional formula** corresponds to a **truth-value function.**
  - Each **truth-value function** corresponds to **infinitely many propositional formulas.**

1-ary Truth-Value Function

p	$F_0^{(1)}$	$F_1^{(1)}$	$F_2^{(1)}$	$F_3^{(1)}$
0	0	0	1	1
1	0	1	0	1

2-ary Truth-Value Function

p	q	$F_0^{(2)}$	$F_1^{(2)}$	$F_2^{(2)}$	$F_3^{(2)}$	$F_4^{(2)}$	$F_5^{(2)}$	$F_6^{(2)}$	$F_7^{(2)}$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

p	q	$F_8^{(2)}$	$F_9^{(2)}$	$F_{10}^{(2)}$	$F_{11}^{(2)}$	$F_{12}^{(2)}$	$F_{13}^{(2)}$	$F_{14}^{(2)}$	$F_{15}^{(2)}$
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

- **Definition 2.13:** Let ***S*** be a set of logical connectives. If any **n-ary (where n≥1)** truth-value function can be represented by a formula containing only the connectives in ***S***, then ***S*** is called a **functionally complete set** (or ***complete set of connectives***).
- **Theorem 2.1:** The following sets of logical connectives are functionally complete.

why negation always include in complete set?

(1) $S_1=\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$

(2) $S_2=\{\neg, \wedge, \vee, \rightarrow\}$ $\quad A\leftrightarrow B \Leftrightarrow (A\rightarrow B)\wedge(B\rightarrow A)$

(3) $S_3=\{\neg, \wedge, \vee\}$ $\quad A\rightarrow B \Leftrightarrow \neg A\vee B$

(4) $S_4=\{\neg, \wedge\}$ $\quad A\vee B \Leftrightarrow \neg\neg(A\vee B) \Leftrightarrow \neg(\neg A\wedge\neg B)$

(5) $S_5=\{\neg, \vee\}$ $\quad A\wedge B \Leftrightarrow \neg(\neg A\vee\neg B)$

(6) $S_6=\{\neg, \rightarrow\}$ $\quad A\vee B \Leftrightarrow \neg(\neg A)\vee B \Leftrightarrow \neg A\rightarrow B$

- **NAND Form:** $p\uparrow q \Leftrightarrow \neg(p\wedge q)$, $\uparrow$ is called the **NAND connective.**
- **NOR Form:** $p\downarrow q \Leftrightarrow \neg(p\vee q)$, $\downarrow$ is called the **NOR connective.**
  - $p\uparrow q$ is **true** if and only if p **and** q **are not both true.**
  - $p\downarrow q$ is **false** if and only if p **and** q **are not both false.**

- Theorem 2.2: $\{\uparrow\},\{\downarrow\}$ are **functionally complete sets of connectives.** (or complete set of connectives).

Proof: $\neg p \Leftrightarrow \neg(p\wedge p) \Leftrightarrow p\uparrow p$

$$p\wedge q \Leftrightarrow \neg\neg(p\wedge q) \Leftrightarrow \neg(p\uparrow q) \Leftrightarrow (p\uparrow q)\uparrow(p\uparrow q)$$

Thus, $\{\uparrow\}$ **is a functionally complete set..**

A similar proof holds for $\{\downarrow\}$.

Objective：

Key Concepts：